wall admittance. Since the admittance at the boundary-layer edge is close to that at the wall for this case, this is not surprising. With increasing injection rate and lower frequency of driving, and the consequent increase in the boundary-layer thickness, the comparison becomes progressively worse as expected.

Another interesting trend shown by the data of Hersh and Walker is that the flow-turning losses deviate from a linear dependence on the injection Mach number as this Mach number is increased. It is precisely under these conditions that the acoustic boundary-layer thickness becomes comparable to the cross-sectional dimensions of the duct. As long as a legitimate acoustic boundary layer is present, the admittance at its edge will vary directly as the injection Mach number, and thus the flow-turning losses will also vary linearly with the injection Mach number. This apparently does not occur when the boundary layer is no longer thin.

Thus, a one-dimensional model for the acoustic wave structure in a duct with side-wall injection is adequate as long as the acoustic boundary-layer thickness is much smaller than the duct cross-sectional dimension and provided that the acoustic admittance at the boundary-layer edge is used. However, to obtain the admittance analytically requires at least a two-dimensional analysis of the boundary-layer region. If the acoustic boundary-layer thickness becomes comparable to the duct cross-sectional dimension, then the acoustic wave structure in the duct will, in general, need at least a two-dimensional treatment to be modeled accurately. The flow-turning losses occur in the acoustic boundary layer so that when it is thin, the losses may be taken to occur close to the side walls. If the acoustic boundary layer is not thin but encompasses a significant portion of the duct, the flow-turning losses are not confined to the near-wall region.

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Stress Analysis of Short Beams

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Introduction

MANY papers have been published on the analysis of short beams. Using Timoshenko's beam theory, the effects of shearing forces and rotatory inertia to the deflections and frequencies of beams were discussed in detail, and Cowper[1] determined the values of shear coefficients k theoretically for various kinds of cross sections.

Taking into account the warping of the section, Levinson[2] introduced the equations of motion, using equilibrium conditions. He indicated that his formula coincides with the one by the Timoshenko's beam theory provided that $k=5/6$ and obtained the deflections and frequencies[3] of the beams.

On the other hand, Murty[4] insisted that, by merely refining the value of shear coefficient, it is not possible to improve the correlation between theory and experiment. By the principle of minimum total energy, he obtained the fundamental equations governing displacements and determined the frequencies and critical loads.[5] But, restrained conditions obtained in the variational calculus are not always satisfied[6] and, strictly speaking, fundamental equations do not satisfy the equilibrium conditions.

Recently, using the results given in the text by Timoshenko and Goodier,[8] Rehfield and Murthy[7] carried out the stress analysis of the beam simply supported at both ends and discussed the effects of transverse shear, nonclassical axial stress and transverse normal strain to the deflections of beams. Later, by his previous method, Murty[9] analyzed the cantilever beam. His results showed that the shearing stresses along the upper and lower edges do not vanish and, moreover, their values become large in the neighborhood of clamped edge.

In the present paper, taking into account the warping of the section, stress analysis is carried out on the short beam subjected to distributed load. Direct stress σ_x in the axial direction is assumed to be in the form of $\Sigma y^i u_i(x)$ and shearing stress τ and transverse direct stress σ_y are determined, using the equilibrium conditions. The fundamental equations governing u_i are introduced by the variational method. What kinds of u_i should be summed is determined by comparing the values of total energy given by each obtained solution.

Fundamental Equations

The case will be considered where the beam simply supported at both ends is subjected to distributed load q. In order to simplify calculus, q is assumed to be constant along the span.

The equilibrium conditions of stresses are expressed as

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau}{\partial y}=0$$

$$\frac{\partial \tau}{\partial x}+\frac{\partial \sigma_y}{\partial y}=0 \tag{1}$$

where σ_x, σ_y and τ are direct stresses in the axial and transverse directions and shearing stress, respectively. Now, σ_x is assumed to be in the form of

$$\sigma_x=\eta u_1+\eta^2 u_2+\eta^m u_m+\eta^u u_n \tag{2}$$

where 2ℓ, $2h$, m, n, and u_i are length and thickness of the beam, odd and even integers and functions with respect to ξ only, and where $\xi=x/\ell$, $\eta=y/h$, and $r=h/\ell$, respectively.

Substituting Eq. (2) into Eq. (1) and integrating with respect to η, τ and σ_y become

$$\tau=-r\left(\frac{\eta^2}{2}u_1'+\frac{\eta^3}{3}u_2'+\frac{\eta^{m+1}}{m+1}u_m'+\frac{\eta^{n+1}}{n+1}u_n'\right)+f(\xi)$$

$$\sigma_y=r^2\left\{\frac{\eta^3}{6}u_1''+\frac{\eta^4}{12}u_2''+\frac{\eta^{m+2}}{(m+1)(m+2)}u_m'' + \frac{\eta^{n+2}}{(n+1)(n+2)}u_n''\right\}-r\eta f'+g(\xi) \tag{3}$$

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where ($'$) denotes the differentiation with respect to ξ and f and g are functions with respect to ξ only. Boundary conditions are

$$\tau=0 \quad \text{at} \quad \eta=\pm 1$$

$$\sigma_y=-q \text{ at } \eta=1, \qquad \sigma_y=0 \text{ at } \eta=-1$$

$$\sigma_x=0, \quad \int_{-1}^{1} \tau d\eta = \pm q\ell/h \quad \text{at } \xi=\pm 1 \tag{4}$$

With the aid of Eqs. (2-4), the following relations will be obtained:

$$u_1=-\frac{3}{m+2}u_m+a, \qquad u_2=-\frac{3}{n+1}u_n$$

$$\frac{f}{r}=\frac{-m+1}{2(m+1)(m+2)}u_m'+\frac{a'}{2}$$

$$\frac{g}{r^2}=\frac{n-2}{4(n+1)(n+2)}u_n''-\frac{q}{2r^2}$$

$$u_i=0 \quad \text{at} \quad \xi=\pm 1 \tag{5}$$

where $a=(\xi^2-1)3q/4r^2$.

Therefore, each stress is expressed in the form of

$$\sigma_x=\left(\eta^m-\frac{3}{m+2}\eta\right)u_m+\left(\eta^n-\frac{3}{n+1}\eta^2\right)u_n+a\eta$$

$$\frac{\tau}{r}=\frac{u_m'}{2(m+1)(m+2)}[-2(m+2)\eta^{m+1}+3(m+1)\eta^2 - (m-1)]+\frac{u_n'}{n+1}(\eta^2-\eta^{n+1})+\frac{a'}{2}(1-\eta^2)$$

$$\frac{\sigma_y}{r^2}=\frac{u_m''}{2(m+1)(m+2)}[2\eta^{m+2}-(m+1)\eta^3+(m-1)\eta] + \frac{u_n''}{4(n+1)(n+2)}[4\eta^{n+2}-(n+2)\eta^4+(n-2)] + \frac{a''}{6}(\eta^3-3\eta-2) \tag{6}$$

As is easily understood, the last condition of Eq. (4) is satisfied automatically.

Total strain energy is

$$\frac{V}{(h\ell/2E)}=\int_{-1}^{1}\int_{-1}^{1}[\sigma_x^2+\sigma_y^2-2\nu\sigma_x\sigma_y+2(1+\nu)\tau^2]d\eta d\xi \tag{7}$$

where E and ν are Young's modulus and Poisson's ratio, respectively. Substitution of Eq. (6) into Eq. (7) and integration with respect to η gives

$$\frac{V}{(h\ell/2E)}=\int_{-1}^{1}(B_0+B_1u_m^2+B_2u_m+B_3u_n^2 + B_4u_m'^2+B_5u_m'+B_6u_n'^2+B_7u_m''^2+B_8u_m'' + B_9u_n''^2+B_{10}u_n''+B_{11}u_mu_m''+B_{12}u_nu_n'')d\xi \tag{8}$$

where B_i are constants except B_5 and B_8. Variational calculus applied to Eq. (8) gives the following fundamental equations and restrained conditions for u_m and u_n.

$$B_7u_m''''+(B_{11}-B_4)u_m''+B_1u_m=(-B_8''+B_5'-B_2)/2$$

$$B_9u_n''''+(B_{12}-B_6)u_n''+B_3u_n=0 \tag{9}$$

$$[(2B_7u_m''+B_{11}u_m+B_8)\delta u_m']_{-1}^{1}$$

$$=[(2B_9u_n''+B_{12}u_n+B_{10})\delta u_n']_{-1}^{1}=0$$

$$\{[2B_7u_m'''+(B_{11}-2B_4)u_m'+B_8'-B_5]\delta u_m\}_{-1}^{1}=0$$

$$\{[2B_9u_n'''+(B_{12}-2B_6)u_n']\delta u_n\}_{-1}^{1}=0 \tag{10}$$

The last two conditions of Eq. (10) are satisfied automatically because $u_m=u_n=0$ at both ends of the simply supported beam. In this situation, u_m and u_n can easily be determined by solving Eq. (9) under the conditions of Eq. (10).

Simplification of Energy Formula and Comparison with Accuracy of Solutions

Although accuracy of solution will be improved by taking more terms of u_i, calculation becomes greatly complicated. So, what kinds of u_i should be assumed will be studied. Accuracy of solution can be investigated by comparing the value of strain energy given by each solution. But, the tedious work of integration will be necessary.

In order to simplify the energy formula, attention will be paid to the similarity in shape between energy formula Eq. (8) and the fundamental Eq. (9). By integration by parts, the following relations will be obtained:

$$\int u_i''u_j''d\xi=u_i''u_j'-u_i'''u_j+\int u_i''''u_jd\xi$$

$$\int u_iu_j''d\xi=u_iu_j'-u_i'u_j+\int u_i''u_jd\xi$$

$$\int u_i'u_j'd\xi=u_i'u_j-\int u_i''u_jd\xi$$

$$\int Bu_i'd\xi=Bu_i-\int B'u_id\xi \tag{11}$$

Rewriting Eq. (8) with the aid of Eq. (11) and substituting Eq. (9) and (10), energy formula V can be simplified in the form of

$$\frac{V}{(h\ell/2E)}=\int_{-1}^{1}u_m[B_7u_m''''+(B_{11}-B_4)u_m''+B_1u_m + B_8''-B_5'+B_2]d\xi+\int_{-1}^{1}u_n[B_9u_n''''+(B_{12}-B_6)u_n'' + B_3u_n]d\xi+\int_{-1}^{1}B_0d\xi+[(B_7u_m''+B_8)u_m']_{-1}^{1} + [(-B_7u_m'''+B_4u_m'-B_8'+B_5)u_m]_{-1}^{1} + [(B_9u_n''+B_{10})u_n']_{-1}^{1}+[(-B_9u_n'''+B_6u_n')u_n]_{-1}^{1} \equiv\int_{-1}^{1}B_0d\xi-A \tag{12}$$

where

$$A=-\int_{-1}^{1}\frac{1}{2}(B_8''-B_5'+B_2)u_md\xi-\left[\frac{B_8u_m'}{2}\right]_{-1}^{1} - \left[\frac{B_{10}u_n'}{2}\right]_{-1}^{1} \tag{13}$$

Since $\int_{-1}^{1}B_0d\xi$ is the value of $V/(h\ell/2E)$ for the case of $\sigma_x=\eta u_1$, it is not necessary to calculate this value. We have

Table 1 Values of $10^2 \times A/q^2$ for various cases of u_i

$1/r$	1	2	3
u_1, u_3	7.5504	4.9140	4.0378
u_1, u_5	7.7408	4.9543	4.0257
u_1, u_0, u_2	17.9573	8.5708	5.6225
u_1, u_2, u_4	17.0245	8.4084	5.6090
u_1, u_0, u_2, u_3	25.5077	13.4920	9.6603
u_1, u_2, u_3, u_4	24.5749	13.3224	9.6468
u_1, u_0, u_2, u_5	25.6981	13.5323	9.6482
u_1, u_2, u_4, u_5	24.7653	13.3627	9.6347

Table 2 Values of $|\sigma_x(u_i)/\sigma_{BE}(\eta=1)|$ at $\xi=0$ for various caes of u_i

	at $\eta=1$			
$1/r$	1.0	1.5	2.0	3.0
u_1, u_0, u_2, u_5	1.7303	1.2058	1.0902	1.0360
u_1, u_0, u_2, u_3	1.6588	1.1799	1.0756	1.0259
u_1, u_5	1.3916	1.1480	1.0814	1.0362
u_1, u_3	1.3201	1.1220	1.0668	1.0296
	at $\eta=-1$			
$1/r$	1.0	1.5	2.0	3.0
u_1, u_0, u_2, u_5	1.0529	1.0901	1.0726	1.0364
u_1, u_0, u_2, u_3	0.9815	1.0642	1.0580	1.0298

Table 3 Distributions of $\sigma_x(u_0, u_1, u_2, u_5)/\sigma_{BE}(\eta=1)$ at $\xi=0$ in the η direction

η	$1/r=1$	$1/r=2$
1.0	1.7303	1.0902
0.8	0.9453	0.8019
0.6	0.4907	0.5748
0.4	0.2016	0.3747
0.2	−0.0076	0.1840
0	−0.1693	−0.0044
−0.2	−0.2892	−0.1917
−0.4	−0.3776	−0.3793
−0.6	−0.4635	−0.5741
−0.8	−0.6339	−0.7938
−1.0	−1.0528	−1.0726

Table 4 Distributions of $\sigma_x(u_0, u_1, u_2, u_5)/\sigma_{BE}(\xi=0)$ along the upper edge

ξ	$1/r=1$	$1/r=2$	$\sigma_x(u_1)/\sigma_{BE}(\xi=0)$
0	1.7303	1.0902	1.0
0.2	1.7027	1.0558	0.96
0.4	1.6193	0.9551	0.84
0.6	1.4248	0.7925	0.64
0.8	0.9907	0.5454	0.36
0.9	0.5996	0.3427	0.19
1.0	0	0	0

only to compare the values of A, that is to say, the larger the value of A, the better the approximation becomes.

Numerical Example and Discussion

Numerical calculations have been carried out for several cases. What kinds of u_i should be summed will be investigated for the case where σ_x is assumed to be in the form of $\eta u_1 + \Sigma \eta^i u_i$. The relationships between the values of A, r, and u_i are listed in Table 1. Accuracy can be improved by taking the term u_n (n is an even integer) for the case where the value of $1/r$ is in the neighborhood of 1. As is easily understood from this table, the most accurate solution can be obtained by summing up u_0, u_1, u_2, and u_5.

The relationships between r and the outer fiber stress at midspan of the beam are listed in Table 2. The term σ_{BE} ($\eta=1$) means the value of outer fiber stress given by the Bernoulli-Euler theory. This is equal to σ_x, which is assumed by u_1 only.

The value of $\sigma(u_0,u_1,u_2,u_5)$ at $\eta=1$, which reaches about 1.7 times σ_{BE} at $1/r=1$, decreases as the value of $1/r$ increases. The values at $\eta=-1$, which are small in comparison with those at $\eta=1$, are nearly equal to σ_{BE} ($\eta=-1$) for all cases.

The distributions of $\sigma(u_0,u_1,u_2,u_5)$ at $\xi=0$ in the η direction are listed in Table 3. Their values become large in the neighborhood of $\eta=1$. The distribution of σ_x at $\eta=1$ in the ξ direction are listed in Table 4. Their values are, in general, large in comparison with σ_{BE} over the span.

In text by Timoshenko and Goodier,[8] analysis is carried out by assuming the stress function to be in the form of $\Sigma x^m y^n$ ($m+n \leq 5$) for the case of the beam simply supported at both ends and subjected to a uniformly distributed load. But, since

$$\int_{-h}^{h} \sigma_x \mathrm{d}y = \int_{-h}^{h} \sigma_x y \mathrm{d}y = 0$$

are used as end conditions, accuracy of solution is low in comparison with that of $\sigma_x(u_1)$ in this paper. As Rehfield and Murthy[7] analyzed, using these results, accuracy of the solutions cannot be expected.

Murty[9] obtained the solution by the variational method and his results do not always satisfy the restrained conditions and equilibrium conditions. Strain energy by σ_y is neglected and Poisson's ratio ν is assumed to be zero. It will easily be understood that energy by σ_y cannot be neglected in comparison with those by σ_x and τ for the case where the value of ℓ/h becomes small. Moreover, as stated previously, shearing stresses do no vanish along the upper and lower edges.

Conclusions

1) In order to improve the accuracy of solution, u_n (n is an even integer) should be taken for the case where the value of ℓ/h becomes small.

2) The value of outer fiber stress σ_x at $\xi=0$ and $\eta=1$, which reaches about 1.7 times σ_{BE} for $\ell/h=1$, decreases as the value of ℓ/h increases, and becomes nearly equal to σ_{BE}, for $\ell/h>4$.

3) On the contrary, the value at $\eta=-1$ is, in general, small and nearly equal to σ_{BE}.

In order to simplify calculus, q is assumed to be constant and the beam simply supported at both ends is treated. But, the case where q is specified by the arbitrary function with respect to ξ can also be analyzed by the present method.

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